

THE RELATIONSHIP BETWEEN DOUBLE- K PARAMETERS OF CONCRETE BASED ON FRACTURE EXTREME THEORY

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The theoretical relationship between the initial fracture toughness and unstable fracture toughness in the double- K model was established based on fracture extreme theory. Using this relationship, the initial fracture toughness and unstable fracture toughness can be obtained from each other without experimental measurement. The values of unstable fracture toughness of three-point bending concrete beams were calculated with the input of initial fracture toughness. The calculation was simplified by using the weight function method. The calculated results were compared with those obtained by the double- K method. It is shown that the results obtained from these two methods show a good agreement. Unstable fracture toughness is affected by tensile strength and specimen size whereas unaffected by the ratio of initial crack length to depth. In addition, the softening function was found to have a negligible influence on the calculated result by a theoretical relationship.

Keywords: initial fracture toughness, unstable fracture toughness, theoretical relationship, simplified extreme method, softening function

1. Introduction

The application of linear elastic fracture mechanics (LEFM) to determine the fracture parameter of concrete was first introduced by Kaplan (1961). Since then, extensive research has been conducted to study the fracture process of quasi-brittle materials. The critical stress intensity factor calculated by LEFM mainly depends on specimen size (Bažant and Planas, 1998). This is due to the nonlinear fracture process zone at the crack tip, and indicates that LEFM is not applicable to concrete. To solve this problem, several nonlinear fracture models have been proposed to describe crack propagation in concrete, such as the fictitious crack model (Hillerborg *et al.*, 1976), crack band model (Bažant and Oh, 1983), two parameter model (Jenq and Shah, 1985), size effect model (Bažant, 1984), effective crack model (Swartz and Go, 1984; Refai and Swartz, 1987; Karihaloo and Nallathambi, 1990), and double- K model (Xu and Reinhardt, 1999a).

The double- K model introduces initial fracture toughness K_I^{ini} and unstable fracture toughness K_I^{un} to characterize three different stages: crack initiation, stable crack propagation, and unstable fracture in concrete. The two fracture parameters can be obtained by performing different laboratory tests and can be applied to evaluate the safety of large-sized concrete structures, such as dams and nuclear power stations. K_I^{ini} represents the capability of resisting crack initiation. The value of K_I^{ini} can be calculated by initial cracking load P^{ini} and initial crack length a_0 using the formula of LEFM (Tada *et al.*, 2000). K_I^{un} represents the maximum resistance to crack propagation, and its value can be calculated by inserting peak load P_{max} and critical

crack length a_c into the same formula of LEFM. a_c is calculated by an LEFM expression based on the linear asymptotic superposition assumption containing critical crack mouth opening displacement $CMOD_c$ (Xu and Reinhardt, 1999b).

According to the double- K method, the relationship between K_I^{ini} and K_I^{un} is established by the critical crack tip stress intensity factor due to the cohesive stress K_I^C , which is also called the cohesive toughness; thus, the following expression (Xu and Reinhardt, 1999b) is given

$$K_I^{ini} = K_I^{un} - K_I^C \quad (1.1)$$

Xu and Reinhardt (2000) adopted two empirical formulae to determine the values of a_c and K_I^C , and thus simplify the calculation. In the simplified method, the special numerical technique to calculate K_I^C in the analytical method (Xu and Reinhardt, 1999b) is not needed. Kumar and Barai (2009) adopted the weight function method and provided a closed form expression for calculating K_I^C . This method reduces the calculation complexity and provides accurate results. On the basis of the weight function method, Ince (2012) applied the peak load method (Tang *et al.*, 1996) to determining the fracture parameters for the double- K fracture model by using splitting test data for different specimen geometries. Kumar *et al.* (2014) proposed a method taking peak load as a parameter to calculate the double- K fracture parameters. In this method, the measurement of $CMOD$ is not required; however, at least three different specimens are suggested to be used for a better estimation. Recently, Qing *et al.* (2017) developed a simplified extreme method using only experimental peak loads to obtain the double- K fracture parameters. This method requires only one specimen to be tested and avoids the numerical integration.

Although the double- K fracture parameters can be obtained from the above-mentioned methods, the theoretical relationship between the two parameters cannot be established due to the requirement of the test. Three-point bending beam tests are difficult to be conducted if the specimens are significantly large, and the influence of specimen weight cannot be ignored. Without the test data, K_I^{un} can be obtained from K_I^{ini} by use of a few analytical methods (Qing *et al.*, 2014; Wu *et al.*, 2015). For example, on the basis of fracture extreme theory (Qing and Li, 2013), Qing *et al.* (2014) adopted the weight function method (Kumar and Barai, 2009) to successfully predict K_I^{un} with the input of K_I^{ini} based on the initial fracture toughness criterion (Dong *et al.*, 2013). In this method, the values of the experimental peak loads and $CMOD_c$ are not required in the calculation due to the application of the extreme value theorem. Comparison of peak loads of wedge splitting specimens validates this approach for predicting the unstable fracture state of concrete. Analytical methods (Qing *et al.*, 2014; Wu *et al.*, 2015) can obtain K_I^{un} from K_I^{ini} but do not illustrate the theoretical relationship between the double- K fracture parameters. Moreover, numerical integration needs to be used in these methods due to the singularity problem at the integral boundary.

The present study aims to establish the theoretical relationship between the double- K fracture parameters based on the simplified extreme method. On the basis of this relationship, the initial fracture toughness and unstable fracture toughness of three-point bending beams can be obtained from each other without experimental measurement. The influences of the ratio of initial crack length to depth, concrete strength grade, and specimen size on the ratio of unstable fracture toughness to initial fracture toughness were studied.

2. Theoretical method

2.1. Fracture extreme theory

Figure 1 shows a typical $P - a/D$ (P is the external load, a is the effective crack length, and D is depth of the specimen) curve of concrete crack propagation. As can be seen, once the external load reaches the initial cracking load P^{ini} , crack begins to propagate. This condition

leads to the extension of crack length. When P reaches the maximum value P_{max} , a equals to critical effective crack length a_c . The partial derivative of P to a at this critical stage can be expressed as

$$\left. \frac{\partial P}{\partial a} \right|_{a=a_c} = 0 \quad (2.1)$$

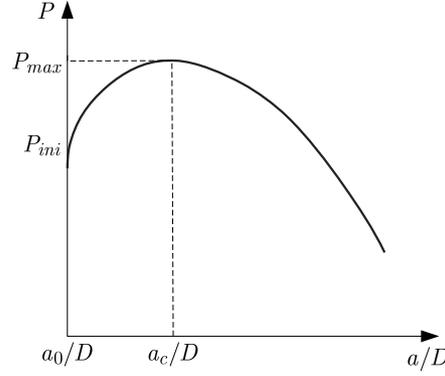


Fig. 1. Typical $P - a/D$ curve (Qing and Li, 2013)

2.2. The relationship between double-K parameters

On the basis of the double-K method (Xu and Reinhardt, 1999b), the values of the initial fracture toughness K_I^{ini} and unstable fracture toughness K_I^{un} for three-point bending beams with $S/D = 4$ (S is the span of the specimen) are calculated as follows

$$K_I = \frac{3(2P + W)S}{4BD^2} \sqrt{a} k(\alpha) \quad (2.2)$$

where B is width of the specimen, $\alpha = a/D$, W is weight of the specimen and

$$k(\alpha) = \frac{1.99 - \alpha(1 - \alpha)(2.15 - 3.93\alpha + 2.7\alpha^2)}{(1 + 2\alpha)\sqrt{(1 - \alpha)^3}}$$

When (P_{max}, a_c) is introduced into Eq. (2.2), the following expression can be obtained

$$P_{max} = \frac{2BD^2}{3S\sqrt{a_c}k(\alpha_c)} K_I^{un} - \frac{W}{2} \quad (2.3)$$

The crack propagation criterion (Dong *et al.*, 2013) is expressed as

$$K_I^{ini} = K_I^P - K_I^\sigma \quad (2.4)$$

where K_I^P is the stress intensity factor caused by the external load; K_I^σ is the crack tip stress intensity factor due to the cohesive stress. Once the difference between K_I^P and K_I^σ equals to K_I^{ini} , the crack begins to propagate.

In this study, numerical integration in the traditional analytical method was avoided by use of the weight function (Kumar and Barai, 2010), which is written as $K_I^\sigma = (2/\sqrt{2\pi a})g(a)$, to calculate the value of K_I^σ . In the said equation, $g(a)$ is expressed in the four-term weight function as

$$g(a) = \sigma(CTOD)a \left(2\sqrt{s} + M_1s + \frac{2}{3}M_2\sqrt{s^3} + M_3s^2 \right) + \frac{f_t - \sigma(CTOD)}{a - a_0} a^2 \left\{ \frac{4}{3}\sqrt{s^3} + \frac{M_1}{2}s^2 + \frac{4}{1}5M_2\sqrt{s^5} + \frac{M_3}{6} \left[1 - \left(\frac{a_0}{a} \right)^3 - 3s \frac{a_0}{a} \right] \right\} \quad (2.5)$$

where $\sigma(CTOD)$ is the cohesive stress at the crack tip; $CTOD$ is the crack tip opening displacement, $s = 1 - a_0/a$, M_1 , M_2 and M_3 can be represented as polynomial expressions of a/D (Kumar and Barai, 2010). Here, the cohesive stress is assumed to be distributed linearly along the cohesive zone. The cohesive stress at the initial crack and crack tip opening displacement $CTOD$ comply with the tensile softening curve, while the cohesive stress at the tip of the effective crack is equal to the tensile strength.

According to Eq. (2.4), the expression of external load P can be derived as

$$P = \frac{2BD^2}{3S\sqrt{ak}(\alpha)} \left[\frac{2}{\sqrt{2\pi a}} g(a) + K_I^{ini} \right] - \frac{W}{2} \quad (2.6)$$

The derivative of P to a is obtained as follows

$$\frac{\partial P}{\partial a} = \zeta'(a) + \eta'(a)K_I^{ini} \quad (2.7)$$

where

$$\begin{aligned} \zeta'(a) &= \frac{4BD^2}{3\sqrt{2\pi}S} \frac{g'(a)k(\alpha)a - g(a)[k'(\alpha)a + k(\alpha)]}{k^2(\alpha)a^2} \\ \eta'(a) &= -\frac{2BD^2}{3S} \left[\frac{\frac{1}{\sqrt{a}}k(\alpha) + \sqrt{a}k'(\alpha)}{2ak^2(\alpha)} \right] \end{aligned} \quad (2.8)$$

$g'(a)$ and $k'(\alpha)$ are given in Appendix.

The following nonlinear softening function (Reinhardt *et al.*, 1986) was used in the calculation

$$\sigma(CTOD) = f_t \left\{ \left[1 + \left(\frac{c_1 CTOD}{w_0} \right)^3 \right] \exp\left(\frac{-c_2 CTOD}{w_0} \right) - \frac{CTOD}{w_0} (1 + c_1^3) \exp(-c_2) \right\} \quad (2.9)$$

where f_t is the tensile strength of concrete; c_1 , c_2 and w_0 are material parameters.

For three-point bending notched beams, the following form (Jenq and Shah, 1985) was used to calculate $CTOD$

$$CTOD = CMOD \sqrt{\left(1 - \frac{a_0}{a}\right)^2 + \left(-1.149 \frac{a}{D} + 1.081\right) \left[\frac{a_0}{a} - \left(\frac{a_0}{a}\right)^2\right]} \quad (2.10)$$

$CMOD$ can be expressed by the formula of LEFM (Tada *et al.*, 2000) for the three-point bending beam with $S/D = 4$ as follows

$$CMOD = \frac{6PSa}{D^2BE} \left[0.76 - 2.28\alpha + 3.87\alpha^2 - 2.04\alpha^3 + \frac{0.66}{(1-\alpha)^2} \right] \quad (2.11)$$

Substituting $P = P_{max}$ and $a = a_c$ into Eqs. (2.9) and (2.10) and combining Eq. (2.3) yields the critical crack tip opening displacement $CTOD_c$ expressed as follows

$$\begin{aligned} CTOD_c &= \left(\frac{4\sqrt{a_c}K_I^{un}}{Ek(\alpha_c)} - \frac{3Sa_cW}{D^2BE} \right) \left[0.76 - 2.28\alpha_c + 3.87\alpha_c^2 - 2.04\alpha_c^3 + \frac{0.66}{(1-\alpha_c)^2} \right] \\ &\cdot \sqrt{\left(1 - \frac{a_0}{a_c}\right)^2 + \left(-1.149 \frac{a_c}{D} + 1.081\right) \left[\frac{a_0}{a_c} - \left(\frac{a_0}{a_c}\right)^2\right]} \end{aligned} \quad (2.12)$$

According to Eqs. (2.9) and (2.12), $\sigma(CTOD_c)$ was rewritten as the function of K_I^{un} and a_c as follows

$$\sigma(CTOD_c) = f_1(K_I^{un}, a_c) \quad (2.13)$$

When P reaches P_{max} , and $a = a_c$, substituting these into Eqs. (2.6) and (2.7) and combining Eqs. (2.12) and (2.13), a nonlinear equation system can be acquired as follows

$$\begin{aligned}
K_I^{un} = & K_I^{ini} + \frac{2}{\sqrt{2\pi a_c}} f_1(K_I^{un}, a_c) a_c \left(2\sqrt{s_c} + M_1 s_c + \frac{2}{3} M_2 \sqrt{s_c^3} + M_3 s_c^2 \right) \\
& + \frac{f_t - f_1(K_I^{un}, a_c)}{a_c - a_0} a_c^2 \left\{ \frac{4}{3} \sqrt{s_c^3} + \frac{M_1}{2} s_c^2 + \frac{4}{15} M_2 \sqrt{s_c^5} + \frac{M_3}{6} \left[1 - \left(\frac{a_0}{a_c} \right)^3 - 3s \frac{a_0}{a_c} \right] \right\} \\
& \frac{4}{\sqrt{2\pi}} [g'(a_c)k(\alpha_c)a_c - g(a_c)k'(\alpha_c)a_c - g(a_c)k(\alpha_c)] \\
& - \left[\sqrt{a_c}k(\alpha_c) + 2\sqrt{a_c^3}k'(\alpha_c) \right] K_I^{ini} = 0
\end{aligned} \tag{2.14}$$

The three unknowns in Eqs. (2.14) are a_c , K_I^{ini} , and K_I^{un} . a_c can be determined by solving the two above-mentioned equations if any one of K_I^{ini} , and K_I^{un} is known. Thus, Eqs. (2.14) provides an implicit expression of the theoretical relationship between K_I^{ini} and K_I^{un} . On the basis of the closed form expression, the values of initial fracture toughness and unstable fracture toughness can be obtained from each other without measurements of $CMOD_c$ and P_{max} . This relationship indicates that the double-K fracture parameters are mutually dependent.

3. Validation and result

The derived theoretical relationship was validated by taking the initial fracture toughness as a parameter to calculate the unstable fracture toughness. If the value of K_I^{ini} is given, Eqs. (2.14) provides two equations with two unknowns of K_I^{un} and a_c . By solving the nonlinear equations, the values of K_I^{un} and a_c can be obtained.

The data of two series of three-point bending notched beams with different ratios of initial crack length to depth (Refai and Swartz, 1987) and different concrete strength grades (Dong *et al.*, 2016; Wang *et al.*, 2016) were used for validation of the theoretical relationship.

Specimens with different ratios of initial crack length to depth were denoted as B -series as in Refai and Swartz (1987). The dimensions for B -series specimens are 76 mm × 203 mm × 760 mm (BDS). The cylinder compressive strength f_c of concrete is 53.1 MPa, and Young's modulus of concrete E is 38.4 GPa, the maximum aggregate size $d_{max} = 19$ mm. The initial fracture toughness K_I^{ini} was calculated by the double- K method.

Specimens with different grades of concrete strength were denoted as TPB-series as in Dong *et al.* (2016) and Wang *et al.* (2016). The dimensions for TPB-series are 60 mm × 120 mm × 480 mm ($B \times D \times S$), and the ratios of initial crack length to depth a_0/D are all 0.3. The tensile strength f_t was computed by using the relation $f_t = 0.4983\sqrt{f_c}$ (ACI-318, 2002; Karihaloo and Nallathambi, 1991). The initial cracking load was obtained by strain gauges, and then the initial fracture toughness was calculated by the formula of LEFM (Tada *et al.*, 2000). Detailed parameters of the specimens are presented in Tables 1 and 2.

Kumar and Barai (2012) studied the size effect of concrete based on finite element method (FEM). The results of three-point bending beams with different specimen sizes were used as an example to validate the developed theoretical relationship. These specimens were denoted as SE-series in this study. The width B of specimens is 100 mm, and $S/D = 4$. The ratios of initial crack length to depth for SE-series specimens are in the range of 0.2 to 0.5. The concrete has mechanical properties as $f_t = 3.21$ MPa and $E = 30$ GPa, and the fracture energy $G_{FC} = 103$ N/m (Planas and Elices, 1990).

The material constants $c_1 = 3$, $c_2 = 7$, and $w_0 = 160$ μ m (Reinhardt *et al.*, 1986) were taken into Eq. (2.9) to calculate K_I^σ .

Table 1. Parameters and comparison of the results of *B*-series specimens ($f_c = 53.1$ MPa, $E = 38.4$ GPa, and $d_{max} = 19$ mm)

Specimen No.	a_0/D	K_I^{ini} [MPa \sqrt{m}]	$CMOD_c$ [μm]		P_{max} [kN]		K_I^{un} [MPa \sqrt{m}]	
			Test result (Refai and Swartz, 1987)	Calculated result	Test result (Refai and Swartz, 1987)	Calculated result	Double- <i>K</i> method (Xu and Reinhardt, 1999b)	Calculated result
B16	0.309	0.51	44	51.6	5.79	6.02	1.358	1.493
B4	0.319	0.507	43.4	52.2	5.612	5.837	1.33	1.482
B17	0.362	0.563	53.3	57.3	5.166	5.31	1.436	1.497
B15	0.376	0.622	43.1	58.5	5.033	5.273	1.264	1.51
B1	0.383	0.795	45.9	59.2	5.523	5.725	1.361	1.576
B3	0.442	0.709	51.1	63.4	4.365	4.503	1.308	1.48
B20	0.459	0.716	61.4	65.3	4.187	4.272	1.43	1.474
B18	0.478	0.766	65.5	67.2	4.053	4.128	1.468	1.478
B19	0.495	0.822	58	69.4	3.919	4.028	1.349	1.491
B5	0.588	0.883	89.5	77.6	3.207	3.31	1.634	1.478
B8	0.636	0.828	80	85.7	2.227	2.287	1.354	1.372
B7	0.648	0.992	89.4	90.4	2.249	2.416	1.457	1.448
B10	0.654	0.818	77.4	88	2.004	2.084	1.286	1.35
B9	0.706	0.843	92.8	94.1	1.537	1.615	1.346	1.407
Avg.	–	–	63.9	70	3.919	4.058	1.384	1.467
S.D.	–	–	18.6	14.6	1.455	1.502	0.095	0.058
C.V.	–	–	29.03%	20.84%	37.12%	37.03%	6.86%	3.93%

Table 2. Parameters and comparison of the results of TPB-series specimens

Specimen No.	Concrete strength grade	f_c [MPa]	E [GPa]	d_{max} [mm]	K_I^{ini} [MPa \sqrt{m}]	P_{max} [kN]		K_I^{un} [MPa \sqrt{m}]	
						Test results (Wang <i>et al.</i> , 2016)	Calculated result	Double- <i>K</i> Method (Dong <i>et al.</i> , 2016)	Calculated result
TPB20-0.3	C20	32.8	29.9	20	0.461	2.485	2.57	1.127	1.031
TPB40-0.3	C40	48.9	33.2	20	0.616	3.283	3.241	1.445	1.275
TPB60-0.3	C60	69.9	35.7	20	0.632	3.501	3.601	1.463	1.445
TPB80-0.3	C80	84.1	38.1	20	0.667	3.806	3.882	1.406	1.588
TPB100-0.3	C100	115.8	41.1	16	0.917	4.761	4.835	1.751	1.866

3.1. Comparison of peak load

The calculated and tested peak loads for B -series (Refai and Swartz, 1987) and TPB-series (Wang *et al.*, 2016) are listed in Tables 1 and 2, respectively. The comparison of peak loads is shown in Fig. 2. As can be seen, the calculated peak loads are slightly higher than the tested ones for the B -series specimens (Refai and Swartz, 1987), while the differences are no more than 5%. For the TPB-series specimens (Wang *et al.*, 2016), the calculated results match the tested results well. The good agreement indicates that unstable fracture toughness can be obtained satisfactorily using the given theoretical relationship.

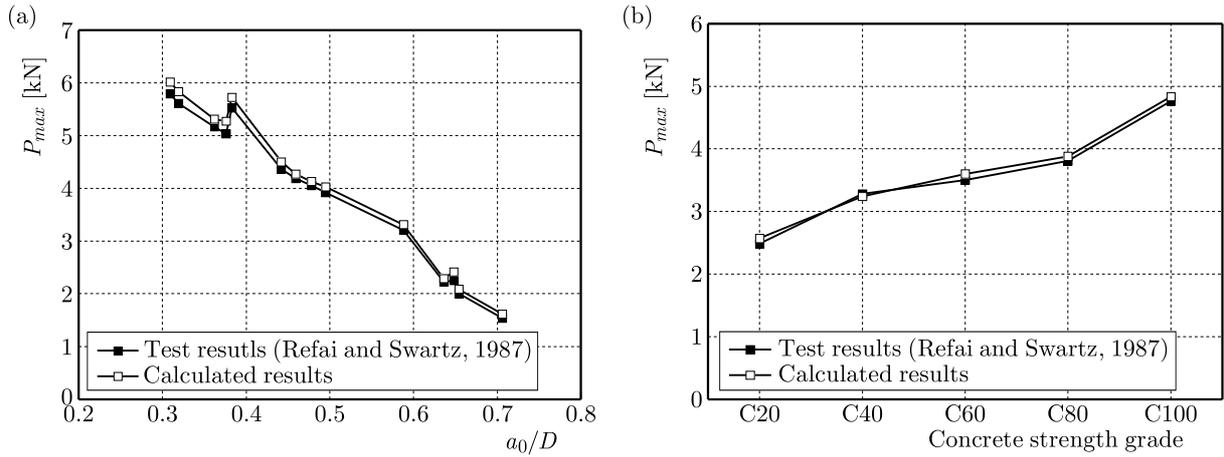


Fig. 2. Comparison of calculated and tested peak loads: (a) B -series, (b) TPB-series

Table 3 shows the comparison of P_{max} obtained using the theoretical relationship with those obtained by FEM. As shown in Table 3, there is not much difference (within the accuracy of 15%) between the values of P_{max} calculated by the theoretical relationship with those calculated by FEM. This means that the method based on the theoretical relationship could produce a reasonable result as the specimen size varies from 100 mm to 400 mm.

3.2. Unstable fracture toughness predicted using the theoretical relationship

Tables 1 and 2 show the calculated results of the unstable fracture toughness K_I^{un} , respectively. In these tables, the values of K_I^{un} in Xu and Reinhardt (1999b) and Dong *et al.* (2016) are obtained by the double- K method. It can be seen that except for specimen “B15”, the values of K_I^{un} obtained by using the theoretical relationship agree with the corresponding values obtained by the double- K method within 15% accuracy. For specimen “B15”, the ratio of the value of K_I^{un} calculated by the theoretical relationship to that by the double- K method is 1.194. Since the values of P_{max} are similar in the two methods, the relatively lower value of K_I^{un} calculated by the double- K method is attributed to the inaccuracy of the tested $CMOD_c$. Compared to the double- K method, the method based on the theoretical relationship is more simple and easier to operate.

Given that the results of K_I^{un} obtained by the theoretical relationship are in good agreement with those by the double- K method, the results of the two methods were chosen for the subsequent analysis. Figure 3a shows the variation of unstable fracture toughness with a_0/D for B -series specimens. It can be seen that the values of unstable fracture toughness vary slightly as a_0/D increases. Therefore, it can be concluded that a_0/D has no significant influence on K_I^{un} . This result is similar to the effect observed by Xu and Reinhardt (1999c).

Table 3. Comparison of the results obtained by FEM and the theoretical relationship ($f_t = 3.21$ MPa, $E = 30$ GPa, and $G_{FC} = 103$ N/m)

Specimen No.	D [mm]	a_0/D	K_I^{ini} [MPa \sqrt{m}]	P_{max} [kN]			K_I^{un} [MPa \sqrt{m}]		
				FEM result (Kumar and Barai, 2012)	Calculated result (nonlinear)	Calculated result (quasi-exponent.)	Double- K method (Kumar and Barai, 2012)	Calculated result (nonlinear)	Calculated result (quasi-exponent.)
SE2-1	100	0.2	0.553	5070.94	5669.85	5808.46	1.224	1.122	1.198
SE2-2	200	0.2	0.547	8502.8	9266.80	9555.46	1.328	1.353	1.395
SE2-3	300	0.2	0.532	11276.59	12392.34	12730.04	1.400	1.541	1.519
SE2-4	400	0.2	0.520	13608.21	15384.23	15627.64	1.419	1.766	1.649
SE3-1	100	0.3	0.572	3934.5	4391.15	4497.42	1.238	1.113	1.185
SE3-2	200	0.3	0.565	6571.4	7128.94	7339.38	1.316	1.331	1.370
SE3-3	300	0.3	0.554	8672.38	9544.79	9766.93	1.377	1.545	1.519
SE3-4	400	0.3	0.539	10405	11819.95	11945.31	1.420	1.763	1.609
SE4-1	100	0.4	0.576	2947.2	3280.91	3358.56	1.212	1.067	1.134
SE4-2	200	0.4	0.576	4909.7	5325.16	5473.15	1.299	1.299	1.335
SE4-3	300	0.4	0.566	6447.9	7116.82	7262.31	1.383	1.503	1.475
SE4-4	400	0.4	0.553	7683.4	8806.06	8873.98	1.416	1.677	1.561
SE5-1	100	0.5	0.575	2095.2	2352.11	2404.43	1.188	1.018	1.079
SE5-2	200	0.5	0.578	3477.2	3803.48	3903.89	1.281	1.232	1.265
SE5-3	300	0.5	0.572	4529.49	5078.35	5177.37	1.351	1.424	1.397
SE5-4	400	0.5	0.562	5335.2	6274.96	6323.69	1.370	1.585	1.507

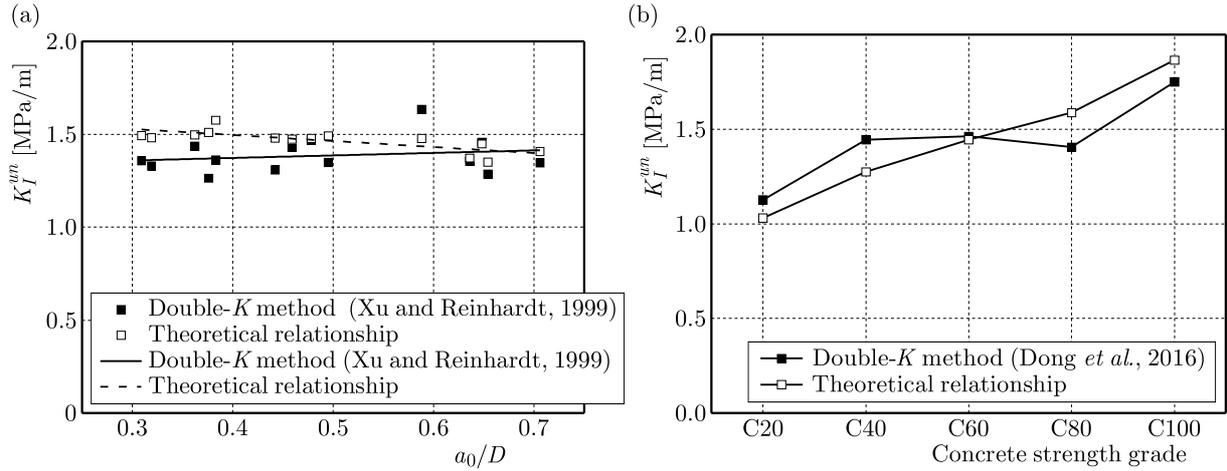


Fig. 3. Variation of unstable fracture toughness with a_0/D (a) and with concrete strength grade (b)

Figure 3b shows the variation of K_I^{un} with the concrete strength grade for TPB-series specimens. It can be noticed that when the concrete strength grade is lower than approximately C60, the values of K_I^{un} calculated by the double-K method are greater than those calculated by the theoretical relationship. However, this tendency is opposite as the concrete strength grade is over C60. In general, the values of K_I^{un} increase with the concrete strength. This result is consistent with that found by Kumar and Barai (2009).

Figure 4 compares the values of K_I^{un} determined by the theoretical relationship with those obtained by FEM (Kumar and Barai, 2012) for SE-series specimens with $a_0/D = 0.3$. It can be seen from Fig. 4 that the values of K_I^{un} obtained by the method based on theoretical relationship are close to those obtained by FEM as the specimen size varies from 100 mm to 400 mm. The unstable fracture toughness shows an increase trend as the specimen size increases. Table 3 indicates that for different values of a_0/D , the variations of K_I^{un} with the specimen size are similar. This means that K_I^{un} is not influenced by a_0/D .

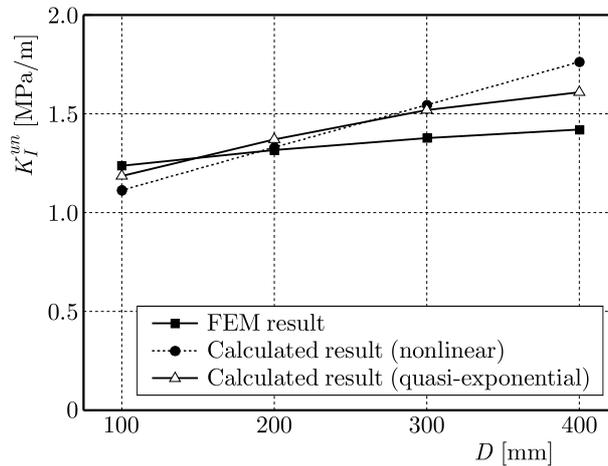


Fig. 4. Comparison of unstable fracture toughness obtained by the theoretical relationship with that obtained by FEM for a specimen of $a_0/D = 0.3$

4. Influence of softening functions

To study the effect of softening functions on the method based on the theoretical relationship, a quasi-exponential softening function (Planas and Elices, 1990) was used to calculate the fracture

toughness of specimens from Kumar and Barai (2012). The quasi-exponential softening function is characterized by the following expression

$$\sigma(CTOD) = \begin{cases} f_t \left[(1 + b_1) \exp\left(\frac{-b_2 w f_t}{G_{FC}}\right) - b_1 \right] & \text{for } 0 < w \leq \frac{5G_{FC}}{f_t} \\ 0 & \text{for } \frac{5G_{FC}}{f_t} \leq w \end{cases} \quad (4.1)$$

$b_1 = 0.0082896 \quad b_2 = 0.96020$

where b_1 and b_2 are material constants.

The calculated results for the peak load and unstable fracture toughness using the quasi-exponential softening function are presented in Table 3. The values of unstable fracture toughness using the quasi-exponential softening function for SE-series specimens with $a_0/D = 0.3$ were added in Fig. 4. It is observed that the results obtained using the quasi-exponential softening function are very close (within the accuracy of 8%) to those obtained using the nonlinear softening. Furthermore, all the calculated results show that the unstable fracture toughness increases with the specimen size. It is evident that concrete shows higher resistance to unstable fracture as the specimen becomes larger.

5. Conclusion

The theoretical relationship between the initial fracture toughness and unstable fracture toughness was established. An implicit closed form expression of the relationship between the two parameters was given. Based on the theoretical relationship, the double- K fracture parameters can be obtained from each other without experimental measurement.

The developed theoretical relationship was validated by the method with the input of the initial fracture toughness to calculate the unstable fracture toughness. Results show that the values of the unstable fracture toughness obtained by the theoretical relationship agree with those obtained by the double- K method and the finite element method. The unstable fracture toughness increases with the specimen size and concrete strength, whereas it shows no appreciable difference when the ratio of initial crack length to depth varies.

The influence of softening functions on the method based on the theoretical relationship is studied. Results show that the calculated values of the peak load and unstable fracture toughness obtained by the nonlinear softening function and the quasi-exponential softening function are close to each other.

A. Appendix

$g'(a)$ and $k'(\alpha)$ in Eqs. (2.8) can be expressed as follows

$$k'(\alpha) = \frac{1}{D(1+2\alpha^2)(1-\alpha)^3} \left\{ (-2.15 + 12.16\alpha - 19.89\alpha^2 + 10.8\alpha^3)(1+2\alpha)\sqrt{(1-\alpha)^3} \right. \\ \left. - (1.99 - 2.15\alpha + 6.08\alpha^2 - 6.63\alpha^3 + 2.7\alpha^4) \left[2\sqrt{(1-\alpha)^3} - \frac{3}{2}(1+2\alpha)\sqrt{(1-\alpha)} \right] \right\} \quad (A.1)$$

$$\begin{aligned}
g'(a) &= (A_1 + A'_1 a) \left(2\sqrt{s} + M_1 s + \frac{2}{3} M_2 \sqrt{s^3} + \frac{1}{2} M_3 s^2 \right) \\
&+ A_1 a \left(\frac{1}{\sqrt{s}} s' + M_1 s' + M'_1 s + M_2 \sqrt{s} s' + \frac{2}{3} M'_2 \sqrt{s^3} + M_3 s s' + \frac{1}{2} M'_3 s \right) \\
&+ A_2 a^2 \left[2\sqrt{s} s' + M_1 s s' + \frac{M'_1}{2} s + \frac{2}{3} M_2 \sqrt{s^3} s' + \frac{4}{15} M'_2 \sqrt{s^5} \right. \\
&+ \left. \frac{M_3}{2} \left(\frac{a_0^3}{a^4} - s' \frac{a_0}{a} + s \frac{a_0}{a^2} \right) \right] + A_2 a^2 \left\{ \frac{M'_3}{6} \left[1 - \left(\frac{a_0}{a} \right)^3 - 3s \frac{a_0}{a} \right] \right\} \\
&+ (2A_2 a + A'_2 a^2) \left\{ \frac{4}{3} \sqrt{s^3} + \frac{M_1}{2} s^2 + \frac{4}{15} M_2 \sqrt{s^5} + \frac{M_3}{6} \left[1 - \left(\frac{a_0}{a'} \right)^3 - 3s \frac{a_0}{a} \right] \right\}
\end{aligned} \tag{A.2}$$

where

$$\begin{aligned}
s' &= \frac{a_0}{a^2} \\
A'_1 &= \frac{\partial \sigma(CTOD)}{\partial a} = \frac{\partial \sigma(CTOD)}{\partial CTOD} \frac{\partial CTOD}{\partial a} \\
A'_2 &= \frac{-\sigma'(CTOD)(a - a_0) - [f_t - \sigma(CTOD)]}{(a - a_0)^2}
\end{aligned} \tag{A.3}$$

for $i = 1$ and 3

$$\begin{aligned}
M'_i &= \frac{1}{\sqrt{(1 - a/D)^3}} \left(\frac{b_i}{D} + 2c_i \frac{a}{D^2} + 3d_i \frac{a^2}{D^3} + 4e_i \frac{a^3}{D^4} + 5f_i \frac{a^4}{D^5} \right) \\
&+ \frac{3}{2D} \frac{1}{\sqrt{(1 - a/D)^5}} \left[a_i + b_i \frac{a}{D} + c_i \left(\frac{a}{D} \right)^2 + d_i \left(\frac{a}{D} \right)^3 + e_i \left(\frac{a}{D} \right)^4 + f_i \left(\frac{a}{D} \right)^5 \right] \\
M'_2 &= \frac{b_i}{D}
\end{aligned} \tag{A.4}$$

According to Eq. (2.9)

$$\begin{aligned}
\frac{\partial \sigma(CTOD)}{\partial CTOD} &= f_t \left\{ \frac{3c_1}{w_0} \exp\left(-\frac{c_2 CTOD}{w_0}\right) \left[\left(\frac{c_1 CTOD}{w_0} \right)^2 \right. \right. \\
&- \left. \left. \frac{c_2}{w_0} \left(1 + \left(\frac{c_1 CTOD}{w_0} \right)^3 \right) \right] - \frac{1}{w_0} (1 + c_1^3) \exp(-c_2) \right\}
\end{aligned} \tag{A.5}$$

Substituting Eq. (2.10) to Eq. (2.9), the expression of $\partial CTOD/\partial a$ can be obtained

$$\begin{aligned}
\frac{\partial CTOD}{\partial a} &= \frac{6PS}{BD^2 E} \left[0.76 - 4.56\alpha + 11.61\alpha^2 - 8.16\alpha^3 + \frac{0.66}{(1 - \alpha)^2} + \frac{1.32\alpha}{(1 - \alpha)^3} \right] \\
&\cdot \left\{ s^2 + (1.081 - 1.149\alpha) \left[\frac{a_0}{a} - \left(\frac{a_0}{a} \right)^2 \right] \right\}^{1/2} \\
&+ \frac{3PSa}{BD^2 E} \left[0.76 - 2.28\alpha + 3.87\alpha^2 - 2.04\alpha^3 + \frac{0.66}{(1 - \alpha)^2} \right] \\
&\cdot \left\{ s^2 + (1.081 - 1.149\alpha) \left[\frac{a_0}{a} - \left(\frac{a_0}{a} \right)^2 \right] \right\}^{-1/2} \\
&\cdot \left\{ 2s s' - \frac{1.149}{D} \left[\frac{a_0}{a} - \left(\frac{a_0}{a} \right)^2 \right] - (1.081 - 1.149\alpha) \left(\frac{a_0}{a^2} - 2 \frac{a_0^2}{a^3} \right) \right\}
\end{aligned} \tag{A.6}$$

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